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Localization length in Dorokhov's microscopic model of multi-channel wires

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Abstract

We derive exact quantum expressions for the localization length L_c for weak disorder in two- and three-chain tight-binding systems coupled by random nearest-neighbour interchain hopping terms and including random energies of the atomic sites. These quasi-1D systems are the two- and three-channel versions of Dorokhov's model of localization in a wire of N periodically arranged atomic chains. We find that, for weak disorder, $L_c^{-1} = N\xi^{-1}$ for the systems considered with $N = (1, 2, 3)$, where ξ is Thouless' quantum expression for the inverse localization length in a single 1D Anderson chain. The inverse localization length is defined from the exponential decay of the two-probe Landauer conductance, which is determined from an earlier transfer matrix solution of the Schrödinger equation on a Bloch basis. Our exact expressions above differ qualitatively from Dorokhov's localization length, identified as the length-scaling parameter in his scaling description of the distribution of the participation ratio. For $N = 3$ we also discuss the case where the coupled chains are arranged on a strip rather than periodically on a tube. From the transfer matrix treatment we also obtain matrices of reflection coefficients, which allow us to find mean free paths and to discuss their relation to localization lengths in the two- and three-channel systems.

1. Introduction

The localization length, L_c, of the quantum states is a fundamental parameter in mesoscopic physics. In particular, for quasi-1D disordered systems (wires) of finite length L, it not only sets the scale beyond which the electron states are effectively localized but it also determines the domain

$$\ell \leqslant L \leqslant L_c, \tag{1}$$

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in which the conductance g_L displays classical ohmic behaviour, $g_L \propto L^{-1}$, corresponding to a diffusive metallic regime. Here ℓ denotes the elastic mean free path and

$$L_c \simeq N\ell, \tag{2}$$

where $N \propto \sqrt{A}$ is the number of scattering channels in a wire of cross-sectional area A. The metallic domain (1) does not exist for 1D chains ($N = 1$), and for real wires with $N \gg 1$ it requires the resistance to be less than some relatively large threshold value. These fundamental results were first established by Thouless [1] and are reviewed in [2, 3].

On the other hand, the notion of scattering channels is itself important. This is because it has permitted the generalization of the well known scaling equation for the evolution of the distribution of resistance (conductance) as a function of length in a 1D chain [4] in terms of the Dorokhov–Mello–Pereyra–Kumar (DMPK) equation [5, 6] for a distribution of scattering parameters related to the conductance in quasi-1D systems. The DMPK equation—together with the many results derived from it—has been reviewed in [7–9].

Despite the important role played by the localization length in multi-channel disordered systems, microscopic analytic studies of it have remained scarce. This is even more surprising, as L_c is actually an intrinsically microscopic quantum parameter of fundamental importance, as recalled above. Note also that a first-principles derivation of a relation between the localization length and the mean free path, such as (2), would require separate calculation of both quantities in a disordered atomistic multi-channel system.

Some years ago Dorokhov [10] discussed a solvable model of multi-channel localization that consisted of N random tight-binding chains coupled by random nearest-neighbour interchain hopping terms and with random site-energies. Dorokhov's aim was to relax the assumption of the isotropy of scattering parameters underlying the derivation of the DMPK equation [5–7] by replacing it with the weaker assumption of equivalent scattering channels [8]. After a fairly sophisticated analysis, which we find difficult to follow, Dorokhov arrives at an evolution equation for the distribution of scattering variables (participation ratio [11]) which involves a single microscopically defined scaling parameter that he identified with the localization length [10]. Dorokhov's expression for the localization length for weak disorder is independent of the number of channels (equation (11) below), which seems surprising. On the other hand, the popular models of transport and localization in multi-channel wires, such as the Thouless tunnel–junction model [1, 2], the random matrix and maximum entropy models [5–8], and the nonlinear sigma model [9], do not address detailed (discrete) microscopic models with specified disorder.

In a recent paper [12]—hereafter referred to as I—the author derived, for weak disorder, exact analytical expressions for localization lengths for two- and three-chain tight-binding systems with random site energies but constant nearest-neighbour interchain (transverse) and intrachain (longitudinal) hopping parameters. In this model the channels are generally non-equivalent, since they are associated with distinct channel wavenumbers in the absence of disorder [12]. The localization length is defined, as usual, by the rate of exponential decay of the conductance. The conductance is determined using a transfer matrix approach for obtaining the amplitude transmission coefficients in the multi-channel Landauer formula.

Motivated by our doubts about the correctness of Dorokhov's result (which, in particular, is incorrect for a 1D chain), we have reconsidered the calculation of the localization length for the case of two and three equivalent channels in his model, using the exact transfer matrix method for weak disorder that was developed in I. In view of the importance of Dorokhov's microscopic model, in the context of scaling theories for probability distributions of transport parameters in quasi-1D systems, it seems important to dispose of an accurate independent description of the localization length. On the other hand, the related analysis of reflection

matrices will allow us to calculate mean free paths for the two- and three-channel systems in the Born approximation and thus to test equation (2).

In section 2 we recall the Schrödinger tight-binding equations for Dorokhov's model for the case of two- and three-chain systems. Dorokhov's model corresponds to periodic boundary conditions (PBCs) for the chains, i.e. it describes equidistant chains that are arranged parallel to the axis on a tube. We also consider an alternative three-chain model with the parallel chains arranged on a planar strip, which corresponds to using free boundary conditions (FBCs) for the chains that are now non-equivalent. In section 3 we summarize the application of the method developed in I for obtaining the explicit forms of the transfer and scattering matrices in the two- and three-chain systems. In section 4 we list and discuss our detailed results for averaged transmission and reflection coefficients as well as for localization lengths and scattering mean free paths. In particular, we allude to a recently studied weakly disordered multi-chain model [13] that includes both interchain and intrachain nearest-neighbour random hopping but no site energy disorder. This model generalizes a well known 1D random hopping tight-binding model in which a de-localization transition has been found at the band centre [14]. We give an exact expression for the localization length in this 1D model, which readily reveals the de-localization transition in the middle of the energy band.

2. Dorokhov's multi-channel model

The N-chain Dorokhov model [10] of a wire consists of parallel linear chains of N_L disordered sites each (of spacing $a = 1$ and length $L = N_L a$) connected at both ends to semi-infinite ideal (non-disordered) chains that constitute the leads. The sites on a given chain, with its associated non-disordered parts, are labelled by integers $1 \leqslant m \leqslant N_L$ in the disordered region and by $m \leqslant 1$ and $m \geqslant N_L$ in the non-disordered regions, respectively. The disordered chains are coupled to each other by random hopping rates (transverse hopping) with vanishing mean values and, correspondingly, the non-disordered chains are decoupled. The system is described by the tight-binding Schrödinger equation:

$$\psi^i_{n+1} + \psi^i_{n-1} + \sum_{j=1}^{N} \varepsilon^{ij}_n \psi^j_n = E\psi^i_n, \qquad i = 1, 2, \ldots, N,\ 1 \leqslant n \leqslant N_L. \quad (3a)$$

$$\psi^i_{n+1} + \psi^i_{n-1} = E\psi^i_n, \qquad n < 1 \text{ or } n > N_L. \quad (3b)$$

Here E is the energy and ψ^i_m denotes the amplitude of the wavefunction at a site m on the ith chain, and $\varepsilon^{ii}_n \equiv \varepsilon^i_n$ is the random energy at a site n on chain i, while

$$\varepsilon^{ij}_n = \varepsilon^{ji}_n, \quad (4)$$

is a random symmetric hopping parameter between a site n on chain i and the corresponding nearest-neighbour site n on chain j. The above energies, including E, are measured in units of a fixed nearest-neighbour matrix element for hopping along the individual chains (longitudinal hopping). The random site energies and hopping parameters are assumed to be identically distributed independent Gaussian variables with vanishing mean and correlation [$\varepsilon_0^2 \equiv (\varepsilon_0)^2$]:

$$\langle \varepsilon^i_n \varepsilon^j_m \rangle = \varepsilon_0^2 \delta_{i,j} \delta_{m,n} \quad (5a)$$

$$\langle \varepsilon^{ij}_n \varepsilon^{pq}_m \rangle = \varepsilon_0^2 \delta_{m,n} (\delta_{i,p}\delta_{j,q} + \delta_{i,q}\delta_{j,p}). \quad (5b)$$

We note that equation (3a) describes a collection of coupled chains of fixed separation, a, arranged parallel to the axis on a tube, which corresponds to PBC for the chains. In the absence of disorder, the chains are independent and equivalent, and (3a) shows that they all couple in the same way to the disorder. Therefore these independent chains define N equivalent scattering channels [10].

We now focus on the specific cases of two- and three-chain systems, which are the object of this paper. For $N = 2$ and 3, equation (3a) may be written as

$$\begin{pmatrix} \psi^1_{n+1} + \psi^1_{n-1} \\ \psi^2_{n+1} + \psi^2_{n-1} \end{pmatrix} = \begin{pmatrix} E - \varepsilon^1_n & -\varepsilon^{12}_n \\ -\varepsilon^{21}_n & E - \varepsilon^2_n \end{pmatrix} \begin{pmatrix} \psi^1_n \\ \psi^2_n \end{pmatrix}, \qquad N = 2, \tag{6}$$

$$\begin{pmatrix} \psi^1_{n+1} + \psi^1_{n-1} \\ \psi^2_{n+1} + \psi^2_{n-1} \\ \psi^3_{n+1} + \psi^3_{n-1} \end{pmatrix} = \begin{pmatrix} E - \varepsilon^1_n & -\varepsilon^{12}_n & -\varepsilon^{13}_n \\ -\varepsilon^{21}_n & E - \varepsilon^2_n & -\varepsilon^{23}_n \\ -\varepsilon^{31}_n & -\varepsilon^{32}_n & E - \varepsilon^3_n \end{pmatrix} \begin{pmatrix} \psi^1_n \\ \psi^2_n \\ \psi^3_n \end{pmatrix}, \qquad N = 3. \tag{7}$$

For the sake of completeness, we also consider, for $N = 3$, the case where parallel chains are arranged on a planar strip that corresponds to FBC. In this case the Schrödinger equation is

$$\begin{pmatrix} \psi^1_{n+1} + \psi^1_{n-1} \\ \psi^2_{n+1} + \psi^2_{n-1} \\ \psi^3_{n+1} + \psi^3_{n-1} \end{pmatrix} = \begin{pmatrix} E - \varepsilon^1_n & -\varepsilon^{12}_n & 0 \\ -\varepsilon^{21}_n & E - \varepsilon^2_n & -\varepsilon^{23}_n \\ 0 & -\varepsilon^{32}_n & E - \varepsilon^3_n \end{pmatrix} \begin{pmatrix} \psi^1_n \\ \psi^2_n \\ \psi^3_n \end{pmatrix}. \tag{8}$$

Clearly, in this case, the channels are non-equivalent but nevertheless well defined.

As in I, we shall determine the inverse localization length from the rate of exponential decay of the conductance of the disordered wires [1, 2, 15]:

$$\frac{1}{L_c} = -\lim_{N\to\infty} \frac{1}{2N} \langle \ln g \rangle, \tag{9}$$

where averaging over the disorder may be used, as usual, because of the self-averaging property of $\ln g$. The conductance is given by the Landauer two-probe conductance formula [2, 3],

$$g = \frac{2e^2}{h} \operatorname{Tr}(\hat{t}\hat{t}^+), \tag{10}$$

where $\hat{t}$ is the transmission matrix whose ij element denotes the amplitude transmitted in channel i at one end of the wire when there is an incident amplitude in channel j at the other end of the wire.

We close this section by recalling the result for the localization length obtained by Dorokhov [10] for an N-channel wire described by (3a). In the notation of equations (3a), (5a) and (5b), this reads

$$L_c = \frac{4 - E^2}{2\varepsilon_0^2}, \tag{11}$$

which is independent of N. This surprising result follows by combining the expression for the localization length—obtained from the scaling equation for the distribution of the participation ratio in the first equality of (6.26) in [10]—with the definitions (2.9), (2.8) and (2.2). Equation (11) will be discussed further in section 4.

3. Transfer matrix analysis

The scattering matrix (S-matrix), which includes the transmission matrix $\hat{t}$ and the corresponding reflection matrix $\hat{r}$ as submatrices, has been derived in I for weak disorder in terms of transfer matrices for coupled two- and three-chain systems. In I the transfer matrix elements were particularized in the case of an Anderson model with a constant nearest-neighbour interchain coupling, in which they were evaluated explicitly to quadratic order in the random site energies. Here we apply a similar analysis to the case of the Dorokhov models (6)–(8).

The first step thus consists of deriving transfer matrices for the Schrödinger equations (6)–(8). Following I, we first define transfer matrices that describe the transfer of plane waves across single-site transversal slices of the multi-chain wires. These matrices—denoted $\hat{Y}_n$ in I, with $\hat{Y}_n \equiv \hat{X}_{0n}$ for $N = 2$ and $\hat{Y}_n \equiv \hat{X}'_n$ and $\hat{Y}_n \equiv \hat{X}''_n$ for $N = 3$ with FBCs and PBCs, respectively—are given by equations (22) ($N = 2$) and (23) ($N = 3$) of I, in terms of generic parameters that allow specialization for both the constant hopping model of I and the models (6)–(8) above. Thus, for the case of the $N = 2$ and 3 Dorokhov models (6)–(8), we find that the parameters in (22) and (23) of I have the following expressions:

$$a_{1n} = \frac{\varepsilon_n^1}{2\sin k}, \qquad a_{2n} = \frac{\varepsilon_n^2}{2\sin k}, \qquad b_n = -\frac{\varepsilon_n^{12}}{2\sin k}, \tag{12}$$

for $N = 2$,

$$\begin{aligned} &a_{1n} = \frac{\varepsilon_n^1}{2\sin k}, \qquad a_{3n} = \frac{\varepsilon_n^3}{2\sin k}, \qquad b_{2n} = \frac{\varepsilon_n^2}{2\sin k}, \\ &c_n = f_n = \frac{\varepsilon_n^{12}}{2\sin k}, \qquad d_n = q_n = \frac{\varepsilon_n^{23}}{2\sin k}, \qquad g_n = p_n = 0, \end{aligned} \tag{13}$$

for $N = 3$ with FBC, and

$$\begin{aligned} &a_{1n} = \frac{\varepsilon_n^1}{2\sin k}, \qquad a_{3n} = \frac{\varepsilon_n^3}{2\sin k}, \qquad b_{2n} = \frac{\varepsilon_n^2}{2\sin k}, \\ &c_n = f_n = \frac{\varepsilon_n^{12}}{2\sin k}, \qquad g_n = p_n = \frac{\varepsilon_n^{13}}{2\sin k}, \qquad d_n = q_n = \frac{\varepsilon_n^{23}}{2\sin k}, \end{aligned} \tag{14}$$

for $N = 3$ with PBC. Here the wavenumber k, which replaces the distinct wavenumbers k_1, k_2 and k_3 in (22) and (23) of I, is given by

$$2\cos k = E, \tag{15}$$

for energies restricted to the band $-2 \leqslant E \leqslant 2$. This wavenumber describes the equivalent scattering channels for the leads since, in the absence of disorder, the tight-binding chains in (6)–(8) are decoupled. For definiteness, we choose $0 \leqslant k \leqslant \pi$, so that the eigenfunctions

$$\psi_{n,\pm}^j \sim \mathrm{e}^{\pm \mathrm{i}nk}, \tag{16}$$

correspond to Bloch waves travelling from left to right and from right to left, respectively.

On the other hand, as shown in I, the transfer matrices of disordered wires of length $L = N_L a$ are the products of Bloch wave transfer matrices associated with the N_L individual thin slices n:

$$\hat{Y}_L = \prod_{n=1}^{N_L} \hat{Y}_n. \tag{17}$$

For weak disorder, it is sufficient to expand (17) to linear order in the random energies ε_n^i and ε_n^{ij} for the purpose of studying averages to the lowest order in the correlations (5*a*) and (5*b*). These correlations indeed imply that different slices in (17) are uncorrelated. The Bloch wave transfer matrices are given explicitly by equations (30) ($N = 2$) and (32) ($N = 3$) of I, with the parameters defined in (12)–(14) above and wavenumbers k_1, k_2, k_3 replaced by k in (15).

The scattering of plane waves (reflection and transmission) at and between the two ends of the random quasi-1D systems is governed by the S-matrix,

$$\hat{S} = \begin{pmatrix} \hat{r}^{-+} & \hat{t}^{--} \\ \hat{t}^{++} & \hat{r}^{+-} \end{pmatrix}, \tag{18}$$

where

$$\hat{t}^{\mp\mp} = \begin{pmatrix} t_{11}^{\mp\mp} & t_{12}^{\mp\mp} & \cdots \\ t_{21}^{\mp\mp} & t_{22}^{\mp\mp} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}, \tag{19}$$

and

$$\hat{r}^{\pm\mp} = \begin{pmatrix} r_{11}^{\pm\mp} & r_{12}^{\pm\mp} & \cdots \\ r_{21}^{\pm\mp} & r_{22}^{\pm\mp} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}. \tag{20}$$

Here $t_{ij}^{++}(t_{ij}^{--})$ and $r_{ij}^{-+}(r_{ij}^{+-})$ denote the transmitted and reflected amplitudes in channel i when there is a unit flux incident from the left (right) in channel j. The left-to-right and right-to-left directions are labelled + and −, respectively. For current conservation it follows that

$$\sum_{j=1}^{N}(|t_{ji}^{\mp\mp}|^2 + |r_{ji}^{\mp\pm}|^2) = 1. \tag{21}$$

The detailed relationship between the S-matrix and the transfer matrix has been discussed in I. A somewhat lengthy calculation yields the explicit forms—equations (46) and (47) ($N = 2$) and (48) and (48a)–(48f) ($N = 3$) of I—of scattering matrices in terms of transfer matrix elements that are themselves defined in terms of general parameters given by (12)–(14) above in the case of Dorokhov's model. These S-matrices readily yield the transmission and reflection submatrices in (18).

4. Results and concluding remarks

The explicit expressions of the transmission and reflection coefficients, $|t_{ij}^{--}|^2$ and $|r_{ij}^{-+}|^2$, in terms of the general parameters that define the transfer matrices in I are given in an appendix in I. By inserting the present parameter values (12) and (14) for the two- and three-channel Dorokhov models in these expressions and averaging over the disorder, using (5a) and (5b), we obtain the following results, which are exact to order ε_0^2:

$$\langle|t_{11}^{--}|^2\rangle = \langle|t_{22}^{--}|^2\rangle = 1 - \frac{3N_L\varepsilon_0^2}{4\sin^2 k}, \tag{22}$$

$$\langle|t_{12}^{--}|^2\rangle = \langle|t_{21}^{--}|^2\rangle = \frac{N_L\varepsilon_0^2}{4\sin^2 k}, \tag{23}$$

$$\langle|r_{ij}^{-+}|^2\rangle = \frac{N_L\varepsilon_0^2}{4\sin^2 k}, \qquad i, j = (1, 2), \tag{24}$$

for $N = 2$,

$$\langle|t_{11}^{--}|^2\rangle = \langle|t_{33}^{--}|^2\rangle = 1 - \frac{3N_L\varepsilon_0^2}{4\sin^2 k}, \tag{25}$$

$$\langle|t_{22}^{--}|^2\rangle = 1 - \frac{5N_L\varepsilon_0^2}{4\sin^2 k}, \tag{26}$$

$$\langle|t_{12}^{--}|^2\rangle = \langle|t_{21}^{--}|^2\rangle = \frac{N_L\varepsilon_0^2}{4\sin^2 k}, \tag{27}$$

$$\langle|t_{23}^{--}|^2\rangle = \langle|t_{32}^{--}|^2\rangle = \frac{N_L\varepsilon_0^2}{4\sin^2 k}, \tag{28}$$

$$\langle|t_{13}^{--}|^2\rangle = \langle|t_{31}^{--}|^2\rangle = 0, \tag{29}$$

$$\langle|r_{11}^{-+}|^2\rangle = \langle|r_{22}^{-+}|^2\rangle = \langle|r_{33}^{-+}|^2\rangle = \langle|r_{12}^{-+}|^2\rangle = \langle|r_{21}^{-+}|^2\rangle = \langle|r_{23}^{-+}|^2\rangle = \langle|r_{32}^{-+}|^2\rangle = \frac{N_L\varepsilon_0^2}{4\sin^2 k}, \tag{30}$$

$$\langle|r_{13}^{-+}|^2\rangle = \langle|r_{31}^{-+}|^2\rangle = 0, \tag{31}$$

for $N = 3$ with FBC, and

$$\langle|t_{11}^{--}|^2\rangle = \langle|t_{22}^{--}|^2\rangle = \langle|t_{33}^{--}|^2\rangle = 1 - \frac{5N_L\varepsilon_0^2}{4\sin^2 k}, \tag{32}$$

$$\langle|t_{ij}^{--}|^2\rangle = \frac{N_L\varepsilon_0^2}{4\sin^2 k}, \qquad i \neq j,\ i, j = (1, 2, 3), \tag{33}$$

$$\langle|r_{ij}^{-+}|^2\rangle = \frac{N_L\varepsilon_0^2}{4\sin^2 k}, \qquad i, j = (1, 2, 3), \tag{34}$$

for $N = 3$ with PBC.

As a check of the explicit results (22)–(34), one readily verifies that, in all cases ($N = 2$, $N = 3$ with FBC and $N = 3$ with PBC), the current-conservation property (21) is obeyed.

Next, by evaluating the averaged traces $\langle \mathrm{Tr}[\hat{t}^{--}(\hat{t}^{--})^+]\rangle$ successively for the three models using equations (22) and (23), (25)–(29) and (32) and (33), respectively, we get

$$\langle \mathrm{Tr}[\hat{t}^{--}(\hat{t}^{--})^+]\rangle = 2 - \frac{N_L\varepsilon_0^2}{\sin^2 k}, \qquad N = 2, \tag{35}$$

$$= 3 - \frac{7N_L\varepsilon_0^2}{4\sin^2 k}, \qquad N = 3 \text{ with FBC}, \tag{36}$$

$$= 3 - \frac{9N_L\varepsilon_0^2}{4\sin^2 k}, \qquad N = 3 \text{ with PBC}. \tag{37}$$

For the inverse localization lengths that are defined in (9) and (10), we then obtain

$$\frac{1}{L_c} = \frac{\varepsilon_0^2}{4\sin^2 k}, \qquad \text{for } N = 2, \tag{38}$$

$$\frac{1}{L_c} = \frac{7\varepsilon_0^2}{24\sin^2 k}, \qquad \text{for } N = 3 \text{ with FBC}, \tag{39}$$

$$\frac{1}{L_c} = \frac{3\varepsilon_0^2}{8\sin^2 k}, \qquad \text{for } N = 3 \text{ with PBC}. \tag{40}$$

These expressions are exact to order ε_0^2 for weak disorder.

It is instructive to compare (38)–(40) with the localization length, ξ, for weak disorder in a one-dimensional chain with random site energies. In this case, Thouless [16] obtained the exact expression

$$\frac{1}{\xi} = \frac{\varepsilon_0^2}{8\sin^2 k}, \tag{41}$$

which has been re-derived in I (see also [17]) using transfer matrices. We observe that the inverse localization lengths for PBC in (38) and (40) take the values $2/\xi$ and $3/\xi$ for $N = 2$ and for $N = 3$, respectively. The constant value (11) obtained by Dorokhov [10] for arbitrary N differs qualitatively from these exact results. Using (15) and (41), Dorokhov's expression may be written as

$$\frac{1}{L_c} = \frac{4}{\xi}, \qquad N \text{ arbitrary}. \tag{42}$$

In fact, the exact expressions (38) and (40), together with the 1D expression (41), suggest that the actual form for the inverse localization length for weak disorder in a range of low N close to $N = 3$ could be

$$\frac{1}{L_c} = \frac{N}{\xi}. \tag{43}$$

We also note, incidently, that in our analysis of the two- and three-channel Dorokhov models the localization length reduces precisely to the 1D result (41) in the limit of no interchain hopping ($\varepsilon_n^{ij} = 0$), as expected. Indeed, for $\varepsilon_n^{ij} = 0$ we have $t_{ij}^{--} = r_{ij}^{-+} = 0$ for $i \neq j$ and, from the explicit expressions of the random transmission coefficients in the appendix of I, we get, using (12)–(14) and (5a),

$$\langle \mathrm{Tr}[\hat{t}^{--}(\hat{t}^{--})^{+}] \rangle = \begin{cases} 2 - \dfrac{N_L \varepsilon_0^2}{2 \sin^2 k}, & N = 2, \\ 3\left(1 - \dfrac{N_L \varepsilon_0^2}{4 \sin^2 k}\right), & N = 3, \end{cases}$$

which both lead to $1/L_c = 1/\xi$.

On the other hand, the above results for reflection coefficients may be used to obtain explicit expressions for mean free paths in the few-channel quasi-1D systems. The mean free path for an N-channel wire is defined by [8, 18]

$$\frac{1}{\ell_N} = \frac{1}{N_L N} \sum_{i,j} \langle |r_{ij}^{-+}|^2 \rangle. \tag{44}$$

We then obtain successively from equations (24), (30), (31) and (34)

$$\frac{1}{\ell_2} = \frac{\varepsilon_0^2}{2 \sin^2 k}, \tag{45}$$

$$\frac{1}{\ell_3} = \frac{7}{12} \frac{\varepsilon_0^2}{\sin^2 k} \qquad \text{(FBC)}, \tag{46}$$

$$\frac{1}{\ell_3} = \frac{3}{4} \frac{\varepsilon_0^2}{\sin^2 k} \qquad \text{(PBC)}. \tag{47}$$

Similarly, in the one-dimensional case one gets, by returning from (41) to the determination of the reflection coefficient,

$$\frac{1}{\ell_1} = \frac{\varepsilon_0^2}{4 \sin^2 k}. \tag{48}$$

The expressions (45)–(48) correspond to the Born approximation of impurity scattering. By comparing (45)–(48) successively with the localization lengths in (38)–(41), we find that, in all cases,

$$L_c = 2\ell_N, \qquad N = 1, 2, 3. \tag{49}$$

We also note that a similar calculation of mean free paths for the two- and three-channel wire models with constant interchain hopping rates discussed in I [12] also leads to equation (49) for $N = 2$ and 3. The localization lengths for the multi-channel systems in I are given by equations (58), (73) and (86) of that reference, respectively, and the corresponding reflection coefficients that enter into (44) above are given by equations (52)–(57), (67)–(72) and (83)–(85) of [12].

Equation (49) for the one-dimensional case coincides with the relation between the localization length and the mean free path derived by Thouless from kinetic theory [19]. Our treatment thus establishes a similar exact relationship for two- and three-channel systems

both for Dorokhov's model and for the model with constant interchain hopping in I. The exact universal expression (49) differs qualitatively from equation (2) discussed earlier, mainly for $N \gg 1$ [1, 2, 6, 7, 9, 10], and does not suggest the existence of a well defined diffusive (metallic) regime, $\ell_N \ll L \ll L_c$, in few-channel systems. We recall that, in the above references, the mean free path is introduced as a fixed length scale beyond which metallic diffusion takes place (when it is not inhibited by localization), in terms of which equation (2) is derived. Our microscopic analysis yields explicit expressions for both localization lengths and mean free paths.

Finally, from (39) and (40) it follows that the difference in transverse boundary conditions for the corresponding three-channel models has only a minor influence on the localization lengths.

The transfer matrix approach discussed in I may also be applied to studying the delocalization transition, which has recently been found at the band centre in weakly disordered multi-chain systems that include both nearest-neighbour inter- and intrachain random hopping terms but no site energy disorder [13]. This delocalization transition already exists in a one-dimensional chain with random hopping, as has been known for some time [14]. In this case, it may readily be revealed by studying the localization length $L_c^{-1} = -\lim_{L\to\infty}(2L)^{-1}\langle \ln |t^{\pm\pm}|^2\rangle$ of the chain. Consider the Schrödinger equation

$$(1+\eta_n)(\psi_{n+1}+\psi_{n-1}) = E\psi_n, \tag{50}$$

where η_n is a Gaussian random nearest-neighbour hopping parameter (with $\langle \eta_m \eta_n\rangle = \eta_0^2 \delta_{m,n}$), measured in units of the non-random hopping parameter. From a transfer matrix analysis of (50) similar to that used for obtaining the transmission coefficient and the corresponding localization length (41) for a weakly disordered Anderson chain [12, 17], we get

$$\frac{1}{L_c} = \frac{\eta_0^2}{2}\frac{\cos^2 k}{\sin^2 k}, \qquad E = 2\cos k. \tag{51}$$

This expression, which is exact to order η_0^2, displays the divergence of the localization length in the middle of the energy band, $E = 0$.

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